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WEIGHT FUNCTIONS FOR SUB-INTERFACE CRACKS

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Abstract—Weight functions are derived for edge cracks and internal cracks in the vicinity of interfaces of dissimilar materials. The first terms of a power series representation are determined by direct adjusting the weight function to reference stress intensity factors and geometric conditions. FE-calculations with constant internal pressure and constant shear on the crack faces provide the necessary stress intensity factor input. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

Complex stresses occur at the interface of two bonded dissimilar materials which are mechanically or thermally loaded. These stresses become singular, especially near the free edges. For joints where one or both materials are ceramic, the failure starts at small cracks in this brittle material. In order to be able to evaluate the failure behaviour, it is necessary to know the stress intensity factors for natural cracks in the vicinity of the interface. Most of the methods of determination of stress intensity factors require a separate calculation for each stress distribution and each crack length. The weight function method developed by Bueckner (1970) simplifies considerably the determination of stress intensity factors. A weight function exists for any crack problem specified by the geometry of the component and a crack type. If this function is known, the stress intensity factor can be obtained by simply multiplying this function by the stress distribution and integrating it along the crack length. Weight functions are widely used for homogeneous bodies but can hardly be found in the literature for cracks in inhomogeneous materials. Approximate weight functions will be derived below by direct adjustment to simple reference loading cases.

2. WEIGHT FUNCTIONS FOR EDGE CRACKS

Edge cracks parallel to an interface between dissimilar elastic materials, Fig. 1, show mixed-mode stress intensity factors even under pure normal stress or pure shear loading. If the crack faces are loaded with the normal stress σ_y, the stress intensity factors are given by

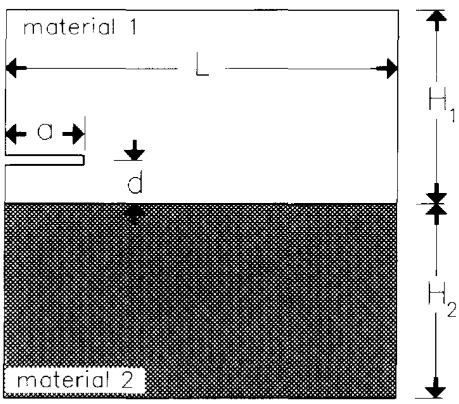

Fig. 1. External subinterface crack ; geometric data.

$$K_I^{(\sigma)} = \int_0^a h_I^{(\sigma)}(x,a)\sigma_y(x)\,\mathrm{d}x, \quad K_{II}^{(\sigma)} = \int_0^a h_{II}^{(\sigma)}(x,a)\sigma_y(x)\,\mathrm{d}x \tag{1}$$

where $h_I^{(\sigma)}$, $h_{II}^{(\sigma)}$ are the weight functions. For shear stresses τ acting at the crack faces one can write

$$K_I^{(\tau)} = \int_0^a h_I^{(\tau)}(x,a)\tau(x)\,\mathrm{d}x, \quad K_{II}^{(\tau)} = \int_0^a h_{II}^{(\tau)}(x,a)\tau(x)\,\mathrm{d}x \tag{2}$$

defining the weight functions as $h_I^{(\tau)}$, $h_{II}^{(\tau)}$. Under combined crack-face loading, the stress intensity factors from eqns (1) and (2) can be superimposed which results in

$$K_I = \int_0^a (h_I^{(\sigma)}(x,a)\sigma_y(x) + h_I^{(\tau)}(x,a)\tau(x))\,\mathrm{d}x \tag{3}$$

$$K_{II} = \int_0^a (h_{II}^{(\sigma)}(x,a)\sigma_{(y)}(x) + h_{II}^{(\tau)}(x,a)\tau(x))\,\mathrm{d}x. \tag{4}$$

The weight functions can be obtained from the stress intensity factors and the displacements of the crack borders in the x- and y-directions of a specific load—the reference load (Rice, 1973). In the case of homogeneous materials the mode-I weight function is related only to the displacements v normal to the crack face and the mode-II weight function to the displacements u in the crack face line. For cracks near the interface we have to expect also an interrelation to exist between the displacements which may be written in a general form

$$h_I^{(\tau)} = m_1 \frac{\partial u^{(\tau)}}{\partial a} + m_2 \frac{\partial u^{(\sigma)}}{\partial a} \tag{5}$$

$$h_I^{(\sigma)} = m_3 \frac{\partial v^{(\tau)}}{\partial a} + m_4 \frac{\partial v^{(\sigma)}}{\partial a} \tag{6}$$

$$h_{II}^{(\tau)} = m_5 \frac{\partial u^{(\sigma)}}{\partial a} + m_6 \frac{\partial u^{(\tau)}}{\partial a} \tag{7}$$

$$h_{II}^{(\sigma)} = m_7 \frac{\partial v^{(\sigma)}}{\partial a} + m_8 \frac{\partial v^{(\tau)}}{\partial a} \tag{8}$$

where the coefficients $m_1, \ldots, m_8$ will depend on the ratio of the Young's moduli E_2/E_1 and Poisson ratios ν_1, ν_2 as well as on the geometric data and the applied load. The indices (τ) and (σ) describe the loadings which are responsible for the crack opening displacements.

2.1. *Set-ups for the weight functions*

For this type of mixed-mode problem the direct adjusting method (Fett, 1992) is an appropriate tool to generate analytical expressions for the four weight function components. Therefore, we will use the following set-ups

$$h_I^{(\sigma)} = \sqrt{\frac{2}{\pi a}} \sum_{\nu=0}^{\infty} D_{I,\nu}^{(\sigma)} (1 - x/a)^{\nu - 1/2} \tag{9}$$

$$h_{II}^{(\sigma)} = \sqrt{\frac{2}{\pi a}} \sum_{\nu=0}^{\infty} D_{II,\nu}^{(\sigma)} (1 - x/a)^{\nu - 1/2} \tag{10}$$

$$h_I^{(\tau)} = \sqrt{\frac{2}{\pi a}} \sum_{\nu=0}^{\infty} D_{I,\nu}^{(\tau)} (1-x/a)^{\nu-1/2} \tag{11}$$

$$h_{II}^{(\tau)} = \sqrt{\frac{2}{\pi a}} \sum_{\nu=0}^{\infty} D_{II,\nu}^{(\tau)} (1-x/a)^{\nu-1/2} \tag{12}$$

with

$$\begin{aligned} D_{I,0}^{(\sigma)} &= D_{II,0}^{(\tau)} = 1 \\ D_{I,0}^{(\tau)} &= D_{II,0}^{(\sigma)} = 0. \end{aligned} \tag{13}$$

Let us assume that the loading cases for constant pressure (σ = const.) and constant shear (τ = const.) directly on the crack faces are known. Then, a sufficient number of conditions are known in order to be able to compute all coefficients for the approximated representation. Here we will restrict ourself to a two-term weight function of the form

$$h_I^{(\sigma)} = \sqrt{\frac{2}{\pi a}} \left(\frac{1}{\sqrt{1-x/a}} + D_{I,1}^{(\sigma)} \sqrt{1-x/a} + D_{I,2}^{(\sigma)} (1-x/a)^{3/2} \right) \tag{14}$$

$$h_{II}^{(\sigma)} = \sqrt{\frac{2}{\pi a}} (D_{II,1}^{(\sigma)} \sqrt{1-x/a} + D_{II,2}^{(\sigma)} (1-x/a)^{3/2}) \tag{15}$$

$$h_I^{(\tau)} = \sqrt{\frac{2}{\pi a}} (D_{I,1}^{(\tau)} \sqrt{1-x/a} + D_{I,2}^{(\tau)} (1-x/a)^{3/2}) \tag{16}$$

$$h_{II}^{(\tau)} = \sqrt{\frac{2}{\pi a}} \left(\frac{1}{\sqrt{1-x/a}} + D_{II,1}^{(\tau)} \sqrt{1-x/a} + D_{II,2}^{(\tau)} (1-x/a)^{3/2} \right). \tag{17}$$

It has been shown for mode-I (Fett *et al.*, 1987) and mode-II loading (Fett, 1990) that the second derivative of the displacements must vanish for the two reference loading cases (σ_y = const., τ = const.). This may be repeated here for the case of the displacements v normal to the crack surface. It has been shown (Fett, 1992) that

$$\begin{aligned} \frac{\partial^2 v}{\partial x^2} &= \frac{2+\nu-\nu^2}{E} \frac{\partial \tau}{\partial x} - \frac{1-\nu^2}{E} \frac{\partial \sigma_x}{\partial y} \\ \frac{\partial^3 v}{\partial x^3} &= \frac{2+\nu-\nu^2}{E} \frac{\partial^2 \tau}{\partial x^2} + \frac{1-\nu^2}{E} \frac{\partial^2 \tau}{\partial y^2}. \end{aligned} \tag{18}$$

In the chosen reference cases, σ_y = const. and τ = const. along the crack faces and thus we obtain, for free surface conditions along the line $x = 0$,

$$\frac{\partial \tau}{\partial x} = 0, \quad \frac{\partial^2 \tau}{\partial x^2} = 0 \quad \forall\, x < a \tag{19}$$

$$\frac{\partial \sigma_x}{\partial y} = 0, \quad \frac{\partial^2 \tau}{\partial y^2} = 0 \quad \text{for } x = 0, \quad \forall\, y \tag{20}$$

and consequently

$$\frac{\partial^2 v}{\partial x^2} = 0, \quad \frac{\partial^3 v}{\partial x^3} = 0 \quad \text{for } x = 0. \tag{21}$$

The same can be shown for the displacements u in the x-direction. Introducing this into eqns (5)–(8) leads to

$$\frac{\partial^2 h_I^{(\sigma)}}{\partial x^2} = \frac{\partial^2 h_{II}^{(\sigma)}}{\partial x^2} = \frac{\partial^2 h_I^{(\tau)}}{\partial x^2} = \frac{\partial^2 h_{II}^{(\tau)}}{\partial x^2} = 0 \quad \text{for } x = 0. \tag{22}$$

Equation (22) leads to four relations between the coefficients D_ν:

$$\sum D_\nu(\nu - 1/2)(\nu - 3/2) = 0. \tag{23}$$

If the reference stress distributions $\sigma_y(x)$ and $\tau(x)$ are constant, eqns (1) and (2) lead to four additional equations:

$$\sum D_\nu \frac{2}{2+\nu} = Y\sqrt{\pi/2} \tag{24}$$

where the geometric functions Y are defined as

$$\begin{aligned} K_I^{(\sigma)} &= \sigma_y\sqrt{a}Y_I^{(\sigma)}, \quad K_{II}^{(\sigma)} = \sigma_y\sqrt{a}Y_{II}^{(\sigma)} \\ K_I^{(\tau)} &= \tau\sqrt{a}Y_I^{(\tau)}, \quad K_{II}^{(\tau)} = \tau\sqrt{a}Y_{II}^{(\tau)}. \end{aligned} \tag{25}$$

From eqns (23) and (24), the coefficients are obtained as

$$\begin{aligned} D_{I,1}^{(\sigma)} &= \frac{5}{4}\sqrt{\frac{\pi}{2}}Y_I^{(\sigma)} - 2, \quad D_{I,2}^{(\sigma)} = \frac{5}{12}\sqrt{\frac{\pi}{2}}Y_I^{(\sigma)} - \frac{5}{3} \\ D_{II,1}^{(\sigma)} &= \frac{5}{4}\sqrt{\frac{\pi}{2}}Y_{II}^{(\sigma)}, \quad D_{II,2}^{(\sigma)} = \frac{5}{12}\sqrt{\frac{\pi}{2}}Y_{II}^{(\sigma)} \\ D_{I,1}^{(\tau)} &= \frac{5}{4}\sqrt{\frac{\pi}{2}}Y_I^{(\tau)}, \quad D_{I,2}^{(\tau)} = \frac{5}{12}\sqrt{\frac{\pi}{2}}Y_I^{(\tau)} \\ D_{II,1}^{(\tau)} &= \frac{5}{4}\sqrt{\frac{\pi}{2}}Y_{II}^{(\tau)} - 2, \quad D_{II,2}^{(\tau)} = \frac{5}{12}\sqrt{\frac{\pi}{2}}Y_{II}^{(\tau)} - \frac{5}{3}. \end{aligned} \tag{26}$$

2.2. *Example of application*

Finite element computations were used to determine the weight functions for edge cracks near an interface. The data were obtained for a Young's modulus ratio $E_1/E_2 = 100$ and $\nu_1 = 0.2$, $\nu_2 = 0.4$. The crack was located in material 1. The resulting geometric functions are plotted in Fig. 2(a) as a function of the distance from the interface normalised on the crack length. As could be expected from Saint Venant's principle, the influence of the different material 1 on the stress intensity factors can be neglected for $d/a > 3$. For constant

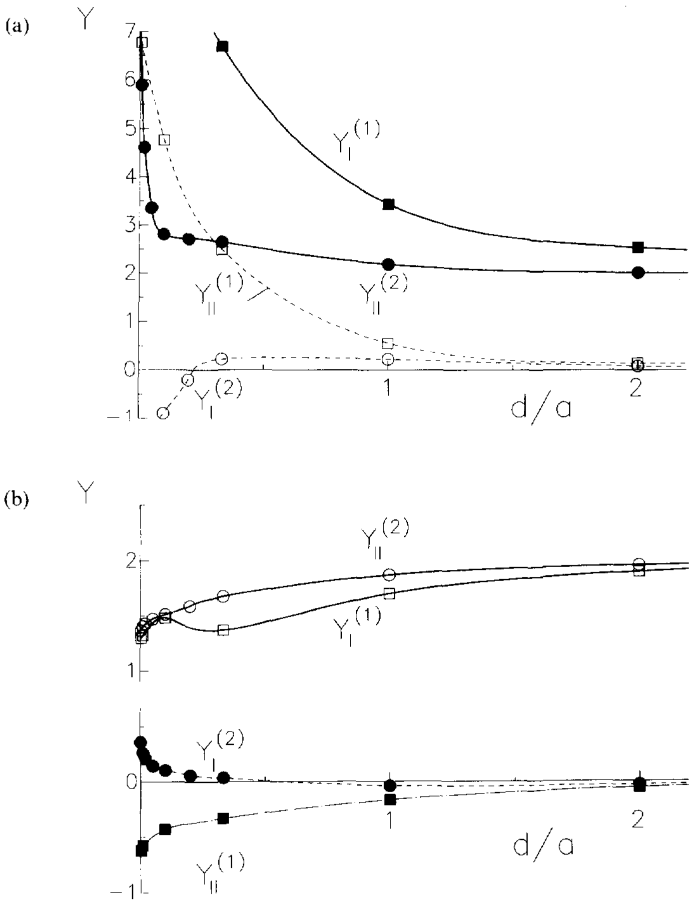

Fig. 2. (a) Geometric function for edge cracks; geometric data: $a/L \leqslant 0.1$, $H_1 = H_2 = 2L$, material data: $E_1/E_2 = 100$, $\nu_1 = 0.2$, $\nu_2 = 0.4$. (b) Geometric function for edge cracks; geometric data: as (a), crack in material 2.

pressure at the crack faces the mode-I stress intensity factor K_I tends against the value of the homogeneous material and K_{II} vanishes. In the case of constant shear stresses at the crack, the mode-II stress intensity factor K_{II} approaches the value of the homogeneous material and K_I vanishes.

The resulting weight functions are plotted in Fig. 3(a) for a crack with $d/a = 1$. Note that the results in Fig. 2(a) and Fig. 3(a) are for a semi-infinite plate or for a finite plate with $a/L \leqslant 0.1$. For the cracks located in material 2, the corresponding results are shown in Figs 2(b) and 3(b).

3. WEIGHT FUNCTIONS FOR INTERNAL CRACKS

The geometry of an internal subinterface crack is illustrated in Fig. 4. The representation of the stress intensity factors by the weight functions are the same as in the case of the subinterface edge crack.

3.1. *Set-ups for the weight functions*

In the special case of *a symmetrically* loaded crack we use the set-ups

$$h_I^{(\sigma)} = \frac{2}{\sqrt{\pi a}}\left(\frac{1}{\sqrt{1-(x/a)^2}} + D_I^{(\sigma)}\sqrt{1-(x/a)^2}\right) \tag{27}$$

$$h_{II}^{(\sigma)} = \frac{2}{\sqrt{\pi a}} D_{II}^{(\sigma)}\sqrt{1-(x/a)^2} \tag{28}$$

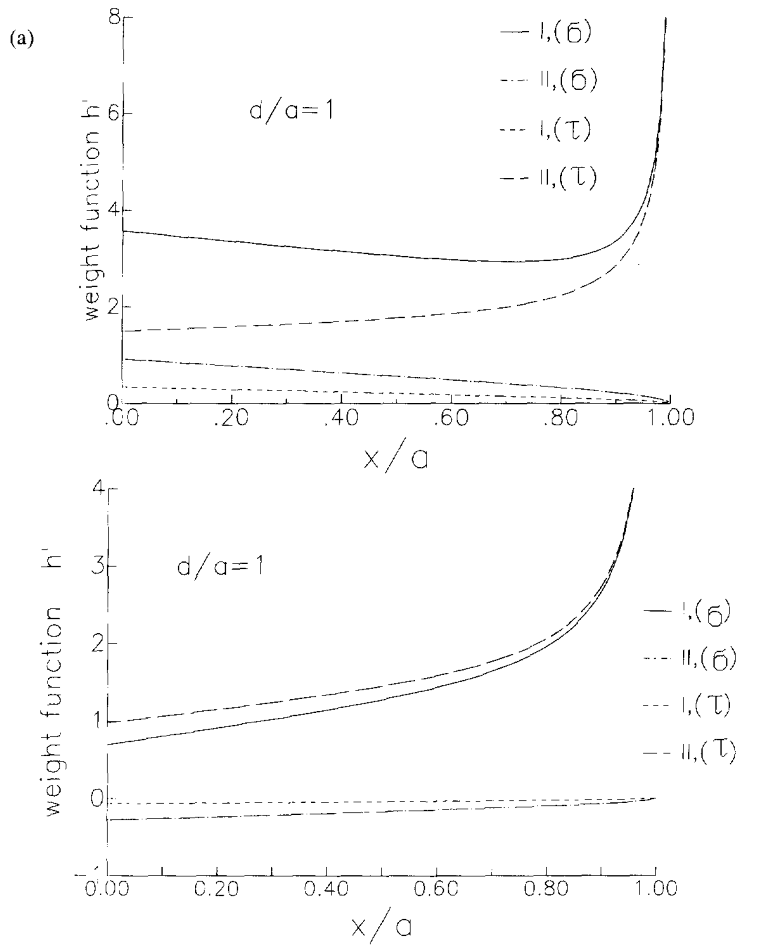

Fig. 3. (a) Weight function for edge cracks. Geometric data : $a/L \leqslant 0.1$, $H_1 = H_2 = 2L$, $E_1/E_2 = 100$, $\nu_1 = 0.2$, $\nu_2 = 0.4$; $d/a = 1$, weight function normalised : $h' = h\sqrt{a}$. (b) Weight function for edge cracks. Geometric data : as (a), crack in material 2.

$$h_I^{(\tau)} = \frac{2}{\sqrt{\pi a}} D_I^{(\tau)} \sqrt{1-(x/a)^2} \tag{29}$$

$$h_{II}^{(\tau)} = \frac{2}{\sqrt{\pi a}} \left(\frac{1}{\sqrt{1-(x/a)^2}} + D_{II}^{(\tau)} \sqrt{1-(x/a)^2} \right). \tag{30}$$

Here only one energy condition is necessary since the set-up fulfills all symmetry conditions

Table 1. Geometric function for subinterface edge-cracks. Geometric data : $a/L \leqslant 0.1$, $H_1 = H_2 = 2L$, $E_1/E_2 = 100$, $\nu_1 = 0.2$, $\nu_2 = 0.4$

d/a	$Y_I^{(\sigma)}$	$Y_{II}^{(\sigma)}$	$Y_I^{(\tau)}$	$Y_{II}^{(\tau)}$
0.0025	10.71	7.290	−4.514	8.6845
0.010	10.93	6.786	−2.838	5.891
0.100	9.952	4.769	−0.898	2.811
0.333	6.682	2.491	0.2165	2.65
1.000	3.437	0.5514	0.2195	2.185
2.000	2.536	0.1384	0.0769	2.0116
3.000	2.273	0.052	0.031	1.986
10.00	2.004	0.002	0.0013	1.974

Table 2. Geometric function for subinterface edge-cracks. Geometric data: as Table 1, crack in material 2

d/a	$Y_I^{(\sigma)}$	$Y_{II}^{(\sigma)}$	$Y_I^{(\tau)}$	$Y_{II}^{(\tau)}$
0.0025	1.298	−0.6203	0.3581	1.370
0.010	1.322	−0.576	0.2602	1.403
0.100	1.483	−0.4316	0.1039	1.513
0.333	1.368	−0.3337	0.0341	1.675
1.000	1.696	−0.1663	−0.0397	1.865
2.000	1.888	−0.05654	−0.0306	1.9486
3.000	1.981	−0.0225	−0.0143	1.965
10.00	1.935	−0.001	−0.0006	1.971

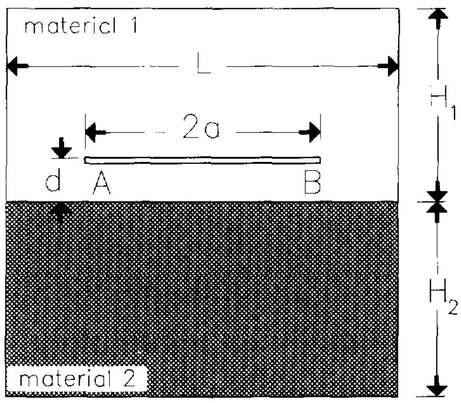

Fig. 4. Internal subinterface crack; geometric data.

a priori. We obtain the coefficients

$$D_I^{(\sigma)} = \frac{2}{\sqrt{\pi}} Y_I^{(\sigma)} - 2 \tag{31}$$

$$D_{II}^{(\sigma)} = \frac{2}{\sqrt{\pi}} Y_{II}^{(\sigma)} \tag{32}$$

$$D_I^{(\tau)} = \frac{2}{\sqrt{\pi}} Y_I^{(\tau)} \tag{33}$$

$$D_{II}^{(\tau)} = \frac{2}{\sqrt{\pi}} Y_{II}^{(\tau)} - 2. \tag{34}$$

3.2. *Example of application*

Internal subinterface cracks under pure tensile and pure shear loading were investigated by Isida and Noguchi (1983) applying the Body Force Method. From their data, we use $H_1 = H_2 \to \infty$, $E_1/E_2 = 4$, $v_1 = v_2 = 0.3$ and $d/2a = 0.2$:

$$Y_I^{(\sigma)} = 2.432, \quad Y_{II}^{(\sigma)} = 0.315, \quad Y_I^{(\tau)} = 0.250, \quad Y_{II}^{(\tau)} = 1.932. \tag{35}$$

The resulting weight function is plotted in Fig. 5.

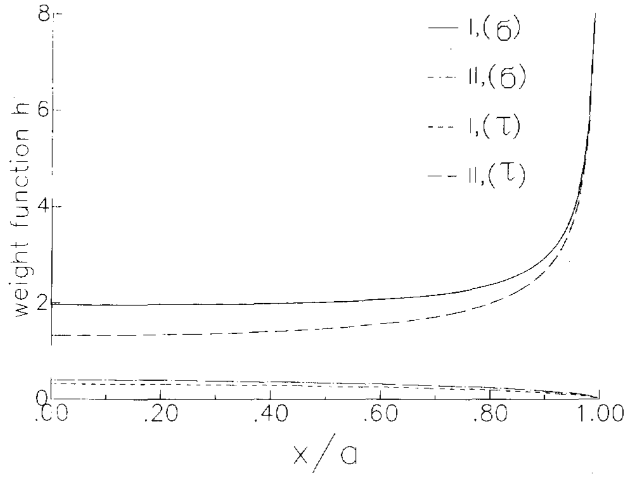

Fig. 5. Weight function for internal cracks. Geometric data : $H_1 = H_2 \rightarrow \infty$, $E_1/E_2 = 4$, $\nu_1 = \nu_2 = 0.3$; $d/2a = 0.2$, weight function normalised : $h' = h\sqrt{a}$.

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